of constants for all 38 waveguides (using the "accepted value of c) is generally available at the present time.

A complete cross-referenced set of tables has recently been completed at this laboratory using the Frederic (Feranti-Mercury) computer. For each of the 38 waveguide sizes, λ (in cm), λ_g (in cm and inches), $1/\lambda_g$ (in cm^{-1} and $inches^{-1}$), λ_g/λ, and λ/λ_g are tabulated against frequency. Entitled "Intern Rapport E-22, 'Tables of Constants for Thirty-Eight Rigid Hollow Metal Rectangular Waveguides,' 12 November 1963," the complete report is available on request from: *Norwegian Defence Research Establishment, Box 25, Kjeller, Norway.*

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The Use of the Rayleigh-Ritz Method in Nonself-Adjoint Problems

This communication is a comment on the very interesting paper[1] by S. P. Morgan in the May issue of these TRANSACTIONS. It may be of interest to point out that problems similar to those discussed by Morgan arise in nuclear reactor theory. Here the operators are not, as in footnote[1], complex symmetric integral operators but they are nonself-adjoint, and there is considerable interest in finding their eigenvalues and eigenvectors by Rayleigh-Ritz methods. Morgan is correct, of course, in pointing out that the usual maximum and minimum criteria are lacking in these cases and that there are no bounds or error estimates. However the conclusion that it is impossible to use the methods may be overly pessimistic. The methods have been used[2-5] in reactor theory with considerable success (measured by comparison with exact solutions), and this fact gives hope that they may be useful in laser applications.

The key point in successful use of Rayleigh-Ritz methods is the selection of appropriate "trial" functions. This is an art which is quickly developed by experience and knowledge of the physical process. In the reactor applications one tactic which has been found very effective is to choose trial functions which in a sense "bound" the true eigenfunction. For example, suppose it is known that the true eigenfunction has a peak in the center but it is not known how high the peak is. One would then choose two trial functions, one having a higher and one a lower peak than expected of the eigenfunction. The Rayleigh-Ritz method with the criterion "make stationary" is used to "blend" the two trial functions in the appropriate proportions. The justification for this procedure is basically empirical—it gives good results. However, a theoretical argument has been advanced[6] which tends to make the process somewhat more palatable. The essence of this argument is to view the variational method as a special case of a more general class of approximation methods, the "weighted residual" methods,[7] and then to show that the variational method is, in a certain sense, the best special case within this class.

In the remainder of this communication we outline the variational or Rayleigh-Ritz process in the general nonself-adjoint case and show the connections to the type of operator used in footnote[1].

Let the letters u, v, etc. denote elements of a function space with a complex inner product (u, v), and let L denote a nonself-adjoint linear operator on this space. Then a variational principle for the eigenvalues of L is that the functional

$$F[u, v] = \frac{(u, Lv)}{(u, v)} \tag{1}$$

be stationary with respect to arbitrary independent variations of the argument functions u and v. If $\hat{u}$, $\hat{v}$ denotes the point where F takes on the stationary value λ, then about this point the first variation is

$$\delta F = \frac{1}{(\hat{u}, \hat{v})}\left[(\delta u, [L-\lambda]\hat{v}) + ([L^*-\bar{\lambda}]\hat{u}, \delta v)\right], \tag{2}$$

so that F is stationary if and only if λ, $\hat{v}$ is an eigenpair of L and $\bar{\lambda}$, $\hat{u}$ is an eigenpair of the adjoint operator L^*.

$$L\hat{v} = \lambda\hat{v}; \qquad L^*\hat{u} = \bar{\lambda}\hat{u}. \tag{3}$$

(Here $\bar{\lambda}$ = complex conjugate of λ.)

To apply the Rayleigh-Ritz process we assume approximate solutions in the form

$$\hat{v} \approx \sum_1^n a_i h_i; \qquad \hat{u} \approx \sum_1^n b_i w_i \tag{4}$$

where the w_i, h_j are known functions and the a_j, b_i unknown parameters. Inserting (4) into (1) yields the ratio of bilinear forms

$$\frac{\begin{bmatrix} \bar{b}_1 & \cdots & \bar{b}_n \end{bmatrix}\begin{bmatrix} & & \\ & (w_i, Lh_j) & \\ & & \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}{\begin{bmatrix} \bar{b}_1 & \cdots & \bar{b}_n \end{bmatrix}\begin{bmatrix} & & \\ & (w_i, h_j) & \\ & & \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}} \tag{5}$$

and requiring this to be stationary yields the matrix eigenvalue problem

$$\begin{bmatrix} & & \\ & (w_i, Lh_j) & \\ & & \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \lambda\begin{bmatrix} & & \\ & (w_i, h_j) & \\ & & \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}. \tag{6}$$

Now we turn attention to the type of operators considered by Morgan.[1] For these operators the adjoint is just the complex conjugate

$$L^* = \bar{L} \tag{7}$$

(where $\bar{L}$ is the operator such that $\bar{L}u = \overline{(Lu)}$ for all u). For such operators (3) shows that the adjoint eigenfunction is the conjugate of the direct eigenfunction

$$\hat{u} = \bar{\hat{v}}. \tag{8}$$

We may make use of this knowledge to specialize the principle (1) in the following way. For operators satisfying (7) the stationary points of $F[u, v]$ have the "natural" property that they are conjugate pairs of functions. Therefore, if the class of admissible pairs (u, v) be restricted to only conjugate pairs, then the functional over the restricted domain becomes

$$F[\bar{v}, v] = \frac{(\bar{v}, Lv)}{(\bar{v}, v)} \tag{9}$$

and has the same stationary points as the unrestricted functional. The functional (9) is recognized as $R[\varphi]$ [(9) in Morgan[1]] by identifying (u, v) with the integral

$$\int_a^b \bar{u}(x)v(x)dx$$

and Lh with the operation

$$\int_a^b k(x, y)h(y)dy.$$

The Ritz process applied to (9) yields

$$\begin{bmatrix} & & \\ & (\bar{h}_i, Lh_j) & \\ & & \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \lambda\begin{bmatrix} & & \\ & (\bar{h}_i, h_j) & \\ & & \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}. \tag{10}$$

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Author's Reply

I wish to thank Dr. Kaplan for calling my attention to the use of variational principles for approximating the eigenvalues of nonself-adjoint operators in nuclear reactor theory. In the problems to which he refers, the functions and operators are all real, and it is possible that variational calculations may be more easily justified for real nonself-adjoint operators than for complex ones.

For complex symmetric operators,[1] it is definitely *not* true that the Rayleigh-Ritz procedure leads to the best approximation obtainable with a given set of trial functions, unless the space spanned by the trial functions happens to include the (unknown) exact eigenfunction. Specifically, it does not minimize the distance between the exact and approximate eigenvalues, and it does not minimize the distance between the exact and approximate eigenfunctions, in terms of a quadratic metric.

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[1] S. P. Morgan, "On the integral equations of laser theory," IEEE TRANS. ON MICROWAVE THEORY AND TECHNIQUES, vol. MTT-11, pp. 191–193; May, 1963.
[2] S. Kaplan, "Some new methods of flux synthesis," *Nucl. Sci. Engrg.*, vol. 13, pp. 22–31; May, 1962.
[3] S. Kaplan, O. J. Marlowe and J. A. Bewick, "Application of synthesis techniques to problems involving time dependence," *Nucl. Sci. Engrg.*, vol. 18, p. 2; February, 1964.
[4] D. S. Selengut, "Variational analysis of multidimensional systems," *Nucl. Phys. Quart. Rept.*, vol. HW-59126, pp. 89–124; January, 1959.
[5] G. P. Calame and F. D. Federighi, "A variational procedure for determining spatially dependent thermal spectra," *Nucl. Sci. Engrg.*, vol. 10, p. 190; 1961.
[6] S. Kaplan, "On the Best Method for Choosing the Weighting Functions in the Method of Weighted Residuals," *Trans. American Nucl. Soc. Mtg.*, Salt Lake City, Utah, vol. 6; June, 1963.
[7] S. H. Crandall, "Engineering Analysis," McGraw-Hill Book Co., Inc., New York, N. Y.; 1956.

As a simple example, let L be a complex symmetric operator and let

$$h = h_1 + \alpha h_2, \tag{1}$$

where h_1 and h_2 are any two trial functions and α is a complex parameter. Then

$$\frac{(\bar{h}, Lh)}{(\bar{h}, h)} = \frac{(\bar{h}_1, Lh_1) + 2\alpha(\bar{h}_1, Lh_2) + \alpha^2(\bar{h}_2, Lh_2)}{(\bar{h}_1, h_1) + 2\alpha(\bar{h}_1, h_2) + \alpha^2(\bar{h}_2, h_2)}. \tag{2}$$

But the right-hand side of (2) is the quotient of two quadratic polynomials in α, and in general it will attain any preassigned value exactly twice as α ranges over the whole complex plane.[8] For example, it can be made equal to any eigenvalue of L, even though h_1 and h_2 are entirely unrelated to the corresponding eigenfunction. On the other hand, applying the Rayleigh-Ritz procedure to h_1 and h_2 yields two values of α for which the corresponding values of $(\bar{h}, Lh)/(\bar{h}, h)$ need not and generally do not coincide with any eigenvalue of L.

A numerical example shows the sort of thing that can happen. Let

$$Lv \equiv \int_{-1}^{1} [1 + i\tfrac{1}{2}(x + y)]v(y)dy, \tag{3}$$

and choose the trial functions

$$h_1(x) = 1 - sx^2 + isx^3, \qquad h_2(x) = 1 + i2x, \tag{4}$$

where s is a real constant to be specified presently. Straightforward calculation gives

$$\begin{aligned}
(\bar{h}_1, Lh_1) &= 4 - \frac{52s}{15} + \frac{32s^2}{45}, \\
(\bar{h}_1, Lh_2) &= \frac{8}{3} - \frac{58s}{45}, \\
(\bar{h}_2, Lh_2) &= \frac{4}{3}, \\
(\bar{h}_1, h_1) &= 2 - \frac{4s}{3} + \frac{4s^2}{35}, \\
(\bar{h}_1, h_2) &= 2 - \frac{22s}{15}, \\
(\bar{h}_2, h_2) &= -\frac{2}{3}.
\end{aligned} \tag{5}$$

We now choose s to be the larger of the two roots of the equation

$$\frac{(\bar{h}_1, Lh_1)}{(\bar{h}_1, h_1)} = 1 + \tfrac{1}{3}\sqrt{6}. \tag{6}$$

To five decimals,

$$s = 1.62669. \tag{7}$$

The determinantal equation resulting from the Rayleigh-Ritz process [Kaplan's (10)] becomes, numerically,

$$\begin{vmatrix} 0.24250 - 0.13350\lambda & 0.57005 + 0.38581\lambda \\ 0.57005 + 0.38581\lambda & 1.33333 + 0.66667\lambda \end{vmatrix} = 0. \tag{8}$$

[8] Notice that the variational quotient associated with a *Hermitian* operator, as given by (4) of Morgan,[1] cannot generally be made to assume arbitrary values by proper choice of α.

The roots of this equation and the corresponding "eigenfunctions" are

$$\tilde{\lambda}_1 = -0.00357, \quad \tilde{v}_1 = h_1 - 0.42727h_2,$$
$$\tilde{\lambda}_2 = -1.91444, \quad \tilde{v}_2 = h_1 + 2.95501h_2. \tag{9}$$

But since the kernel of the operator L is of finite rank, it is easy to calculate the eigenvalues and eigenfunctions exactly. They are[9]

$$\begin{aligned}
\lambda_1 &= 1 + \tfrac{1}{3}\sqrt{6} \approx 1.81650, \\
v_1 &= 1 + i(3 - \sqrt{6})x \approx 1 + i0.55051x; \\
\lambda_2 &= 1 - \tfrac{1}{3}\sqrt{6} \approx 0.18350, \\
v_2 &= 1 + i(3 + \sqrt{6})x \approx 1 + i5.44949x.
\end{aligned} \tag{10}$$

Comparing (6) and (10), we see that the eigenvalue λ_1 coincides exactly with the Rayleigh quotient $(\bar{h}_1, Lh_1)/(\bar{h}_1, h_1)$, even though h_1 is not a multiple of v_1. However the Rayleigh-Ritz procedure gives no indication of this, and when applied to the trial functions h_1 and h_2 it yields "eigenvalues" $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ which are much worse than we would have obtained from h_1 alone. Hence the procedure clearly does not give the best approximation to λ_1 which can be had from a linear combination of h_1 and h_2.

This innocent-looking example shows that the use of the Rayleigh-Ritz procedure to refine an approximate eigenvalue of a complex symmetric operator can actually lead to a worse approximation than the initial one. One might conjecture, of course, that even though the variational method does not approximate eigenvalues very well, it does give an optimal approximation for the eigenfunctions. Unfortunately this is not true either. If one defines the distance between two complex-valued functions in terms of a quadratic metric, then the best approximation to v_1 which can be obtained using a linear combination of h_1 and h_2 is found by minimizing

$$\begin{aligned}
I(\rho_1, \rho_2) &= (v_1 - \rho_1 h_1 - \rho_2 h_2, v_1 - \rho_1 h_1 - \rho_2 h_2) \\
&= \int_a^b |v_1 - \rho_1 h_1 - \rho_2 h_2|^2 dx
\end{aligned} \tag{11}$$

with respect to the complex coefficients ρ_1 and ρ_2. The minimization is easily carried out in the present example where v_1 is known, but it does not lead to either of the Rayleigh-Ritz "eigenfunctions."

Dr. Kaplan suggests that the key to the successful use of variational procedures in non-self-adjoint problems is the selection of appropriate trial functions, in the light of experience and knowledge of the physical process. One can hardly quarrel with this objective, although it may be easier to carry out when the unknown functions are real than when they are complex. Certainly no law prohibits anyone from formally applying the Rayleigh-Ritz procedure to a selected set of functions and thinking that he has obtained a better approximation by so doing. Undoubtedly in some cases he actually will get a better approximation than he started with; but if he does not know the exact eigenvalue in advance, he can never be quite sure whether the Rayleigh-Ritz procedure has served him well or ill.

[9] Zero is also an eigenvalue of L of infinite multiplicity, but it plays no role in the present argument.

To the best of my knowledge, no theorems have ever been proved which specify the conditions, if any, under which the Rayleigh-Ritz procedure will yield an improved approximation to an eigenvalue of a non-self-adjoint (for example, complex symmetric) operator. There *are* such theorems for self-adjoint or Hermitian operators. Whether anything similar can be obtained for nonself-adjoint operators is an interesting open question.

I should like to acknowledge a stimulating exchange of correspondence on this subject with Professor R. F. Harrington of Syracuse University.

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Comment on "Broadband Microwave Discriminator"

The frequency discriminator described by R. J. Mohr[1] is capable of a simple extension[2] to give a device with important advantages and many practical applications.

Mohr's circuit (Fig. 1) measures frequency in the form

$$\frac{|E_2|^2}{|E_1|^2} = \tan^2 \pi l \frac{f}{c}$$

where l is the length of the phase delay line, f is the frequency, c the phase velocity and

$$\frac{2\pi lf}{c} = \phi.$$

An alternative method of processing the detected signals is to take the difference

$$|E_1|^2 - |E_2|^2 = E^2 \cos 2\pi l \frac{f}{c},$$

and by duplicating the circuit so that $\phi' = \phi - (\pi/2)$, we have

$$|E_3|^2 - |E_4|^2 = E^2 \sin 2\pi l \frac{f}{c}$$

and hence

$$\frac{|E_3|^2 - |E_4|^2}{|E_1|^2 - |E_2|^2} = \tan 2\pi l \frac{f}{c} = \tan \phi.$$

Simply, a $\lambda/4$ length of line may be used to subtract $\pi/2$ from the phase delay ϕ and Fig. 2 shows a typical circuit. Some nonlinearity in the frequency characteristic will result from the fact that the phase change of $\pi/2$ will vary with frequency However, this is small and circuits operating over frequencey ranges up to 6:1 in the band 0.15 Gc to 11.5 Gc have been successfully used. An absolute measuring accuracy of $\pm 5°$ in ϕ is typical and may be improved by calibrating individual circuits.

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[1] R. J. Mohr, "Broadband microwave discriminator," IEEE Trans. on Microwave Theory and Techniques, vol. 11, p. 263; July, 1963.
[2] S. J. Robinson, "Microwave Frequency Measuring Device," British Patent Application, No. 22471/58; July, 1958.